Discontinuous generalized synchronization of chaos

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Dedicated to the memory of Jaroslav Stark

Abstract

We study synchronization functions in basic examples of discontinuous forced systems with contractive response and chaotic driving. The forcing is given by baker-type maps and the response is assumed to depend monotonically on the drive. The resulting synchronization functions have dense sets of discontinuities and their graphs appear to be extremely choppy. We show that these functions have bounded variation when the contraction is strong, and conversely, that their total variation is infinite when the contraction becomes weak. In the first case, we also analyze in detail smoothness properties of the corresponding continuous component.

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1 Introduction

Analyzing the asymptotic response to a random or erratic stimulus is a ubiquitous problem in Nonlinear Dynamics. The archetypical example is given by synchronization phenomena in directionally coupled systems [1]. (For convenience, focus will be on discrete time dynamics throughout this paper). When an autonomous forcing $x^{t+1} = f(x^t)$ compels the iterations of a dissipative factor $z^{t+1} = g(x^t, z^t)$, the dynamics is known to be attracted by the invariant graph $z = h(x)$ of an associated synchronization function h [2, 3, 4]. In brief, the response z is asymptotically locked (*viz.* conjugated) to the drive x.

In this context, the regularity and smoothness of h - which depend on f and g - prescribe those drive features (such as *e.g.* Lyapunov exponents, fractal dimensions, etc) that are conveyed to the factor. For instance, Lipschitz regularity implies some estimates on the attractor's dimension. Applications range from modeling of (low-pass) filters in signal analysis [5, 6, 7, 8, 9] to damage detection in material science [10].

The mathematical theory of synchronization functions is part of the broader study of inertial manifolds in dynamical systems. It goes back to the seminal work of Hirsch, Pugh and Shub [11, 12] who proved existence and continuity under the assumption that f is a homeomorphism and g is contracting for z. They also evaluated the Hölder continuity and showed Lipschitz regularity when the contraction of g is stronger than the stronger contraction rate of f (and additional mild assumptions on f and g). These results were perfected later on [13, 14, 15], especially by Stark [16, 17] who established smoothness and extended them to non-uniform contractive responses (see also [18, 19] for recent developments).

Studies of continuity have been pursued beyond the homeomorphic case, not only when the response function g remains blind to drive discontinuities, but also when f is not invertible [20, 21]. However, continuity may not always hold in applications [6, 7, 8] and sensitive response functions also have to be considered.

To that aim, this paper considers discontinuous synchronization graphs in basic examples of skew-products (f, g) with chaotic driving f and contractive response g *i.e.*

$$|g(x, z) - g(x, z')| \leq \lambda |z - z'| \tag{1}$$

where $0 < \lambda < 1$. As noticed in the literature [14, 16, 17], the response h lacks monotonicity and may have infinitely many discontinuity points that generate a dense subset in phase space of the forcing. The appropriate notion to investigate in this case is the overall graph "length" (*i.e.* total variation of h), together with properties of the continuous and discontinuous components. We proceed to a thorough analytic study in the case where f is an extension of the baker's map and $g(x, z) = \lambda z + (1 - \lambda)x$ is linear [6, 8, 14]. The total variation is shown to possibly diverge depending on the contraction parameter λ. When this quantity remains finite, an analysis of regularity and of the derivative of the continuous component is given. The specific form of h here allows for results well beyond the standard theory of real functions.

The basic properties of the synchronization function however do not depend on the piecewise affine nature of the dynamical system under study. In the last section, we present results that hold for more general discontinuous skew-product systems (f, g).

2 Piecewise linear skew-products and synchronization graphs

As announced before, the autonomous forcing in this study materializes via the generalized baker's map $(x,y) \mapsto f(x,y) = (T_{a,b}(x,y), T_b(y))$ of the unit square $[0,1]^2$ into itself [6, 8]. Here $0 < a, b < 1$ and the mapping components write

$$T_{a,b}(x,y) = \left\{ \begin{array}{ccc} ax & \text{if} & 0 \leq y < b \\ (1-a)x + a & \text{if} & b \leq y \leq 1 \end{array} \right.$$

and

$$T_b(y) = \left\{ \begin{array}{ccc} \frac{y}{b} & \text{if} & 0 \leq y < b \\ \frac{y-b}{1-b} & \text{if} & b \leq y \leq 1 \end{array} \right.$$

Remark: These notations are generic and will often be used with other parameters/variables throughout the text. For instance, $T_a(x)$ denotes the interval map above with parameter a and variable x (instead of b and y respectively).

Due to the contraction $0 < \lambda < 1$ the response system $z^t \mapsto z^{t+1} = g(x^t, z^t) = \lambda z^t + (1-\lambda)x^t$ results to be asymptotically locked to the forcing term [2, 3, 4]. More precisely, the large time behavior of the sequence $\{z^t\}$ is independent of z^0 and approaches $\{h(x^t)\}$ where the synchronization function h is given by

$$h(x) = (1-\lambda) \sum_{t=0}^{\infty} \lambda^t T_a^{t+1}(x), \quad \forall x \in [0,1]. \tag{2}$$

This property is a consequence of the equality $z^t - h(x^t) = \lambda^t (z^0 - h(x^0))$. The response h only depends on the first forcing variable because f is invertible and the first coordinate of the inverse $f^{-1}(x,y) = (T_a(x), T_{b,a}(y,x))$ only depends on x. (Besides, as a function of λ, the map h is smooth and strictly increasing and uniformly converges to T_a when $\lambda \to 0$.) Furthermore, this function solves the conjugacy equation

$$g(x, h(x)) = h \circ T_{a,b}(x,y), \quad \forall x, y \in [0,1]^2.$$

thanks to the property $T_a \circ T_{a,b}(x,y) = x$. When f is not invertible, h *a priori* depends on backward histories $\{x^t\}_{t \leq 0}$, see *e.g.* [20, 21].

Exponential convergence of the series (2) guarantees that h is well-defined on $[0,1]$. The function can be viewed as a uniform limit of piecewise affine maps obtained by truncating the series to finite order (see equation (4) in the proof of Proposition 3.2 below). This comment provides a convenient way to numerically compute the graph of h with arbitrary accuracy, see Figure 1.

The map T_a is right continuous and piecewise increasing. The same properties hold for all maps T_a^t. By uniform convergence, right continuity

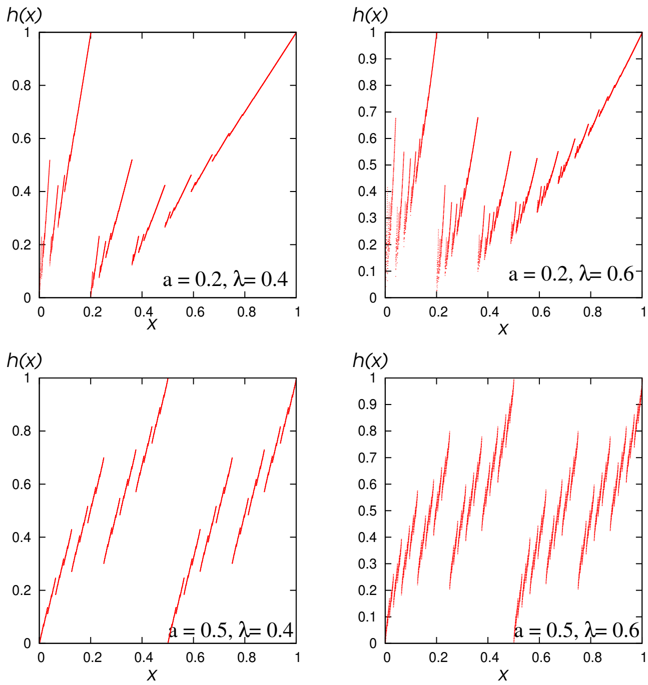

Figure 1: Examples of graph of h (of the function h_{14} indeed - see equation (4)). In the left pictures ($\lambda < 1/2$), h is of bounded variation; in the right ones ($\lambda > 1/2$), it has infinite variation. Notice the symmetry $h(x) = 1 - h(1 - x - 0)$ for $a = 0.5$ (bottom pictures).

transfers to h, *viz.* we have $h(x+0) = h(x)$ for all x. In addition, the maps T_a^t are all piecewise increasing; thus the left limit $h(x-0)$ exists everywhere. This limit coincides with $h(x)$ unless x is a pre-image of the discontinuity point a. More precisely, we have

$$h(x-0) > h(x) \quad \text{iff} \quad x \in D_a := \bigcup_{t=0}^{\infty} T_a^{-t}(a).$$

Since T_a is expanding, D_a is a dense subset of $[0,1]$. To prove that all jump discontinuities are negative, we start to notice that uniform convergence yields

$$h(x-0) = (1-\lambda)\sum_{t=0}^{\infty} \lambda^t T_a^{t+1}(x-0), \quad \forall x \in [0,1].$$

By definition, for every point $x \in D_a$ there is a unique $t_0 \geq 0$ such that $T_a^{t_0}(x) = a$ and $T_a^t(x) \neq a$ when $0 \leq t < t_0$. The map T_a is continuous everywhere but at the point a; hence we have

$$T_a^{t+1}(x-0) = T_a^{t+1}(x), \quad \forall 0 \leq t < t_0.$$

Using again $T_a^{t_0}(x) = a$, we get $T_a^{t_0+1}(x-0) = 1$ and $T_a^{t_0+1}(x) = 0$. However, these points 0 and 1 are fixed points; hence the same values hold for subsequent iterates. Altogether we obtain explicit estimates for jump discontinuities at every point of D_a

$$h(x-0) - h(x) = (1-\lambda)\sum_{t=t_0}^{\infty} \lambda^t > 0.$$

On the other hand, if x lies outside D_a, we have $\lim\limits_{y\to x} T^t(y) = T^t(x)$ for all $t \geq 0$. Uniform convergence then implies that $\lim\limits_{y\to x} h(y) = h(x)$ as claimed.

Beside negative jump discontinuities, the function h has positive increments in arbitrary small left-neighborhoods of every point outside D_a. More precisely, for every $x \in D_a^c$ and $\epsilon > 0$ there exists $y \in (x-\epsilon, x)$ such that $h(y) < h(x)$. Therefore, h is 'nowhere monotonous' (*i.e.* there is no interval on which h is monotonous) and its graph must be extremely wrinkled and choppy as Figure 1 indicates.

Proof that for every $x \in D_a^c$, there exists $y < x$ and arbitrarily close such that $h(y) < h(x)$. Given $x \in D_a^c$ and $\epsilon > 0$, let

$$t_0 = \min\{t \geq 0 \ : \ \exists\, n \ : \ x - \epsilon < x_t^n < x\},$$

where x_t^n is defined by $T_a^t(x_t^n) = a$. The number t_0 exists because D_a is dense. By definition of t_0, all iterates of $x_{t_0}^n$ and x up to $t_0 - 1$ must lie on the same side of the discontinuity point. By strict monotonicity of T_a^t this implies $T_a^t(x_{t_0}^n) < T_a^t(x)$ when $0 \leq t \leq t_0$. Moreover, we have $T_a^t(x_{t_0}^n) = 0$ when $t > t_0$ and $T_a^t(x) > 0$ for all t since $x \in D_a^c$. It easily follows that $h(x_{t_0}^n) < h(y)$. $\square$

Finally, uniform convergence and right continuity allow one to prove that the range of h is an entire interval, the unit interval indeed because of normalization, *i.e.* we have $h([0,1]) = [0,1]$, see Figure 1.

Proof that Ran$(h) = [0,1]$. The crucial point is to show that for every n we have $h_n([0,1]) = [0, 1-\lambda^{n+1}]$ where the approximations h_n are defined in the relation (4) below.

Consider the cylinder sets $[\theta^0 \cdots \theta^n]$ that are associated with the usual symbolic dynamics of $(T_a, [0,1])$, *i.e.* $\theta^t \equiv H(T_a^t(x) - a)$ where H is the Heaviside function. Cylinder sets are intervals and their union (with length fixed) covers $[0,1]$.

The iterates T_a^{t+1} are piecewise bijections and we have $T_a^{t+1}([\theta^0 \cdots \theta^n]) = [\theta^{t+1} \cdots \theta^n]$ for $0 \leq t \leq n$ (where $[\theta^{n+1} \cdots \theta^n]$ should be understood as $[0,1]$). This yields

$$h_n([\theta^0 \cdots \theta^n]) = (1-\lambda) \sum_{t=0}^{n} \lambda^t [\theta^{t+1} \cdots \theta^n]$$

which is an interval. Moreover, given two adjacent cylinders $[\theta^0 \cdots \theta^n] \leq [\bar{\theta}^0 \cdots \bar{\theta}^n]$, their images $[\theta^{t+1} \cdots \theta^n]$ and $[\bar{\theta}^{t+1} \cdots \bar{\theta}^n]$ are either equal or adjacent. As a result, the intervals $h_n([\theta^0 \cdots \theta^n])$ and $h_n([\bar{\theta}^0 \cdots \bar{\theta}^n])$ must intersect; hence $h_n([0,1])$ must be an interval. Computing its extrema gives $h_n([0,1]) = [0, 1-\lambda^{n+1}]$.

The previous property immediately implies that $\mathrm{Ran}(h)$ is dense in $[0,1]$. Indeed for any strict sub-interval I in this set, there must be at least one point of $h([0,1])$. To see this, it suffices to take n such that $|h(x) - h_n(x)|$ is uniformly smaller than $|I|/2$. Assume also that n is sufficiently large so that I is contained in $[0, 1-\lambda^{n+1}]$. Choose x such that $h_n(x)$ is the middle of I. Then $h(x)$ must belong to I.

Now, by continuity we know that every $h(x)$ for $x \in D_a^c$ can be realized as a limit $\lim\limits_{y \to x} h(y)$. For points in D_a, we similarly take right limits. This shows that $h([0,1])$ is the right closure of a dense subset in $[0,1]$, namely it consists of this entire interval excepted the right boundary 1. But $h(1) = 1$, thus the proof is complete. $\square$

3 Total variation and component regularity of the response function

With such an irregular graph for the synchronization function, the appropriate characteristic to evaluate is the total variation [22]; this quantity

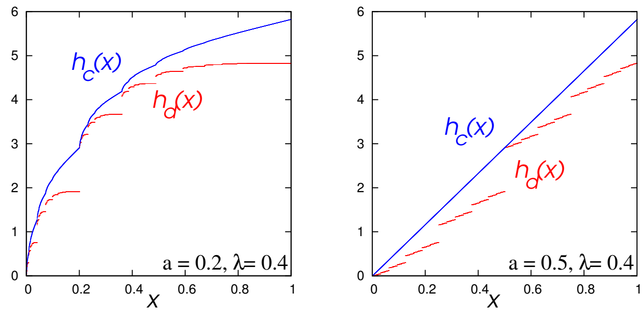

Figure 2: Examples of graphs of h_c (blue dots - upper continuous curve) and h_d (red dots - lower discontinuous curve) when h is of bounded variation ($\lambda = 0.4$); Left $a = 0.2$; Right $a = 0.5$.

measures the integral of the modulus of the derivative. As formally claimed in the next statement, the finiteness of the total variation depends on the contraction factor λ. When the variation is finite, we also simultaneously provide the decomposition into the difference $h_c - h_d$ of two increasing functions and the one into the sum $h_c + (-h_d)$ of a continuous function h_c and a step function $-h_d$ (both decompositions are granted by standard theorems on BV-functions [22]). Some illustrations are given in Figure 2.

Theorem 3.1 *h is of bounded variation iff $\lambda < \frac{1}{2}$. Under this condition, the function h_d given by*

$$h_d(x) = \sum_{y \in D_a \ : \ y \leq x} h(y-0) - h(y), \quad \forall x \in [0,1]$$

is well-defined (and is increasing) and the function h_c defined by $h_c = h + h_d$ is continuous and strictly increasing on $[0,1]$ with range $[0, \frac{2-2\lambda}{1-2\lambda}]$.

Proof. We begin to prove that the variation of h is infinite when $\lambda \geq \frac{1}{2}$. Since all discontinuities of h are negative jumps, the total variation is at least $\sum_{x \in D_a} h(x-0) - h(x)$. There are 2^n points in D_a for which $T_a^n(x) = a$ and the corresponding difference $h(x-0) - h(x)$ is equal to $(1-\lambda) \sum_{t=n}^{\infty} \lambda^t$. Therefore the variation of h is at least

$$\sum_{x \in D_a} h(x-0) - h(x) = \sum_{n=0}^{\infty} (2\lambda)^n. \tag{3}$$

It easily follows that the total variation is infinite when $\lambda \geq \frac{1}{2}$.

To continue, we assume that $\lambda < \frac{1}{2}$. Relation (3) implies that the function h_d is well-defined (increasing and bounded) on $[0,1]$. Moreover $h_d(x-0) = \sum_{y \in D_a \ : \ y < x} h(y-0) - h(y)$, hence $h_d(x) - h_d(x-0) = h(x-0) - h(x)$.

All discontinuities of h are contained in h_d and thus the map h_c defined by $h_c = h + h_d$ is continuous.

We now show that h_c is strictly increasing and bounded, and as a consequence, that h is of bounded variation. Assume that $x < y$ and using the same notations as in the proof above that h is nowhere monotonous, let

$$t_0 = \min\{t \geq 0 \ : \ \exists\, n \ : \ x < x_t^n \leq y\},$$

where x_t^n is defined by $T_a^t(x_t^n) = a$. We have

$$T_a^t(x) < T_a^t(x_{t_0}^n) \leq T_a^t(y), \quad 0 \leq t \leq t_0.$$

Hence

$$h(x) < (1-\lambda)\left(\sum_{t=0}^{t_0-1} \lambda^t T_a^{t+1}(x_{t_0}^n) + \sum_{t=t_0}^{\infty} \lambda^t\right) = h(x_{t_0}^n - 0)$$

and

$$(1-\lambda)\sum_{t=0}^{t_0-1} \lambda^t T_a^{t+1}(x_{t_0}^n) = h(x_{t_0}^n) \leq h(y)$$

(The sum from 0 to $t_0 - 1$ equals 0 if $t_0 = 0$.) Using also the definition of h_d and h_c, we obtain

$$\begin{array}{rcl} h_c(x) & = & h(x) + h_d(x) \\ & < & h(x_{t_0}^n - 0) + h_d(x) \\ & \leq & h(x_{t_0}^n - 0) + h_d(x_{t_0}^n - 0) = h_c(x_{t_0}^n), \end{array}$$

and

$$\begin{array}{rcl} h_c(y) & = & h(y) + h_d(y) \\ & \geq & h(x_{t_0}^n) + h_d(y) \\ & \geq & h(x_{t_0}^n) + h_d(x_{t_0}^n) = h_c(x_{t_0}^n), \end{array}$$

i.e. $h_c(x) < h_c(y)$. Finally we have $h_c(0) = 0$ and $h_c(1) = h(1) + h_d(1) < +\infty$ and the proof is complete. □

We now study in detail smoothness properties of the continuous component h_c. In a general setting, Kolmogorov's theorem says that monotonicity implies the existence of a finite derivative almost everywhere. In our case, the derivative $h_c'(x)$ of the continuous component h_c turns out to exist everywhere outside the countable set D_a, provided that λ is further restricted.

Proposition 3.2 *If $\lambda < \min\{a, 1-a\}$, then the map h_c is differentiable with bounded derivative at any point in the complement D_a^c. The derivative can be written*

$$h_c'(x) = (1-\lambda)\sum_{t=0}^{\infty} \lambda^t (T_a^{t+1}(x))', \quad \forall x \in D_a^c.$$

In addition for any $\lambda \geq \min\{a, 1-a\}$, there are points in D_a^c where $h_c'(x)$ diverges.

Proof. The map h can be regarded as the uniform limit of the sequence $\{h_n\}$ where

$$h_n(x) = (1-\lambda)\sum_{t=0}^{n} \lambda^t T_a^{t+1}(x), \quad \forall x \in [0,1]. \tag{4}$$

Each map h_n is right continuous and with negative jump discontinuities in the set $D_a^n = \bigcup_{t=0}^{n} T_a^{-t}(a)$. As expected, the map $h_{n,d}$ defined by

$$h_{n,d}(x) = \sum_{y \in D_a^n \,:\, y \leq x} h_n(y-0) - h_n(y), \quad \forall x \in [0,1]$$

is an increasing step function that contains all discontinuities of h_n. Consequently, the subsequent map $h_{n,c} = h_n + h_{n,d}$ is a piecewise affine continuous map with finitely many affine parts. As such, it is absolutely continuous and hence differentiable almost everywhere in $[0,1]$ [22]. The derivative $h_{n,c}'$ is a summable function. The fundamental theorem of calculus then yields

$$h_{n,c}(x) - h_{n,c}(0) = \int_0^x h_{n,c}'(y)dy, \quad \forall x \in [0,1] \tag{5}$$

The map $h_{n,c}$ is actually differentiable on $[0,1] \setminus D_a^n$. Letting $T_a'(a) = \frac{1}{1-a}$, the derivative $h_{n,c}'$ can be uniquely continued to the following step function on $[0,1]$

$$h_{n,c}'(x) = (1-\lambda)\sum_{t=0}^{n} \lambda^t (T_a^{t+1}(x))'$$

Since $T_a'(x) \in \{\frac{1}{a}, \frac{1}{1-a}\}$, when $\lambda < \min\{a, 1-a\}$, the sequence $\{h_{n,c}'\}$ uniformly converges to a bounded map, say h_c'. Applying Lebesgue's dominated convergence theorem, one can take the limit $n \to \infty$ in (5) to obtain the following equality

$$h_c(x) - h_c(0) = \int_0^x h_c'(y)dy, \quad \forall x \in [0,1]$$

Now, the map h_c' is uniformly approximated by step functions and is continuous at every point of $[0,1] \setminus D_a$. A standard result [23] states that the map $x \mapsto \int_0^x h_c'(y)dy$ is differentiable at every point of $[0,1] \setminus D_a$ and with derivative h_c'. We conclude from the previous equality that h_c is differentiable on $[0,1] \setminus D_a$ with derivative h_c'. $\square$

The function h_c can not be differentiable in D_a when $a \neq \frac{1}{2}$ because the right and left derivatives of T_a are unequal at $x = a$. However, by artificially setting $T_a'(a) = \frac{1}{1-a}$ for the derivative at the discontinuity point, the domain of h'_c extends to the entire interval $[0,1]$ when $\lambda \leq \min\{a, 1-a\}$. Alternatively, the map h'_c can be regarded as the (well-defined) right derivative on $[0,1]$. The extended map h'_c intriguingly shares several properties with the original function h. Assuming $a \neq \frac{1}{2}$, this map is right continuous with jump discontinuities at every point in D_a. (This map is constant when $a = \frac{1}{2}$.) Unlike for h however, the signs of h'_c jumps do depend on parameters. The analysis reveals that all quantities $h'_c(\cdot - 0) - h'_c(\cdot)$ are positive if $(a - \frac{1}{2})(\lambda - a(1-a)) > 0$ and negative otherwise. Moreover we have the following statement analogous to Theorem 3.1.

Proposition 3.3 *Assume $a \neq \frac{1}{2}$. The function h'_c is of bounded variation iff $\lambda < a(1-a)$. Under this condition, the map $(h'_c)_d$ defined by*

$$(h'_c)_d(x) = \sum_{y \in D_a \,:\, y \leq x} h'_c(y - 0) - h'_c(y)$$

is well-defined on $[0,1]$ *(and non-increasing or non-decreasing depending on the sign of $a - \frac{1}{2}$).*

In addition the function defined by $(h'_c)_c = h'_c + (h'_c)_d$ is continuous and strictly increasing if $a < \frac{1}{2}$ (resp. strictly decreasing if $a > \frac{1}{2}$). It is differentiable at any $x \in D_a^c$ and the derivative identically vanishes.

Proof. To begin, each map $x \mapsto (T_a^{t+1}(x))'$ is piecewise constant and right continuous; hence h'_c is also right continuous. As before, the set D_a collects all discontinuity points. A similar calculation to that in the proof of (2) shows that if $x \in D_a$ is such that $T_a^{t_0}(x) = a$ for some $t_0 \geq 0$, then we have

$$h'_c(x - 0) - h'_c(x) = C_{a,\lambda} \lambda^{t_0} (T_a^{t_0}(x))'$$

where

$$C_{a,\lambda} = (1-\lambda)\left(\frac{1}{a(1 - \frac{\lambda}{1-a})} - \frac{1}{(1-a)(1-\frac{\lambda}{a})}\right).$$

In particular, the sign of $h'_c(x-0) - h'_c(x)$ is independent of x. However, it depends on parameters via the constant $C_{a,\lambda}$. The latter is positive iff $(a - \frac{1}{2})(\lambda - a(1-a)) > 0$.

Now in order to estimate the total variation, as before, we have to sum up all contributions from discontinuities. Given x such that $T_a^{t_0}(x) = a$ for some $t_0 \geq 0$, the derivative $(T_a^{t_0}(x))'$ can be written $a^{-k}(1-a)^{-(t_0-k)}$ where k is the number of those iterates $\{T_a^t(x)\}_{t=0}^{t_0-1}$ that are smaller than a. Up to few sequences, the symbolic dynamics of T_a is the full shift on two symbols. Thus, for every $0 \leq k \leq t_0$, there are $\binom{t_0}{k}$ points with k iterates lying below a. Consequently, the total variation of $h'_c / C_{a,\lambda}$ is bounded below by

$$(1-\lambda)\sum_{t=0}^{\infty} \lambda^t \sum_{k=0}^{t} \binom{t}{k} \frac{1}{a^k (1-a)^{t-k}} = \frac{1-\lambda}{\lambda} \sum_{t=0}^{\infty} \left(\frac{\lambda}{a(1-a)}\right)^{t+1}$$

which is infinite when $\lambda \geq a(1-a)$.

To prove bounded variation when $\lambda < a(1-a)$, we again proceed as before. Assume that $a < \frac{1}{2}$ and consider the non-decreasing step function $(h'_c)_d$ associated with h'_c, *viz.*

$$(h'_c)_d(x) = \sum_{y \in D_a \,:\, y \leq x} h'_c(y-0) - h'_c(y), \quad \forall x \in [0,1]$$

and also the continuous component defined by $(h'_c)_c = h'_c + (h'_c)_d$. Similar arguments to those in the proof of Theorem 3.1 show that the map $(h'_c)_c$ is strictly increasing and bounded on $[0,1]$. It follows that the variation of h'_c is finite when $\lambda < a(1-a)$. An analogous construction applies when $a > \frac{1}{2}$.

As in the proof of Proposition (3.2), the proof that $(h'_c)_c$ is differentiable proceeds by considering a sequence of uniform approximations of h'_c by piecewise affine functions with finitely many branches. $\square$

4 Extensions to Nonlinear Systems

The basic results on synchronization functions in piecewise monotonous forced systems actually do not rely on the affine assumption. They extend to more general systems. To present extensions, we start by introducing nonlinear generalized baker's maps with finitely many pieces.

Let $N > 1$ be an arbitrary integer and consider the finite collections $\{I_i\}_{i=1}^N$ and $\{J_i\}_{i=1}^N$ of intervals defined by

$$I_i = [x_i, x_{i+1}) \text{ where } 0 = x_1 < x_2 < \cdots < x_{N+1} = 1$$

and

$$J_i = [y_i, y_{i+1}) \text{ where } 0 = y_1 < y_2 < \cdots < y_{N+1} = 1$$

for $i = 1, \cdots, N-1$, together with $I_N = [x_N, 1]$ and $J_N = [y_N, 1]$. Consider two mappings T and S on $[0,1]$ defined by

$$T|_{I_i} \equiv T_i \quad \text{and} \quad S|_{J_i} \equiv S_i, \quad i = 1, \cdots, N$$

where each $T_i : I_i \to [0,1)$ (and $T_N : I_N \to [0,1]$) is a C^1 increasing, one-to-one and onto function and similarly for S_i. Now define the map f on $[0,1]^2$ by $f(x,y) = (\mathcal{T}(x,y), S(y))$ where

$$\mathcal{T}(x,y) = T_i^{-1}(x) \text{ if } y \in J_i$$

The map f is easily checked to be invertible with inverse given by $f^{-1}(x,y) = (T(x), \mathcal{S}(y,x))$ where

$$\mathcal{S}(y,x) = S_i^{-1}(y) \text{ if } x \in I_i$$

As in the piecewise affine case, the dynamics of the skew-product (f, g) (where $g(x, z) = \lambda z + (1 - \lambda)x$ still remains unchanged) is attracted by the graph of a function h defined by equation (2) where T_a is replaced by T.

The new synchronization function h shares several properties with the original one. It is piecewise increasing, right continuous with negative jump discontinuities at every point of the set D defined by

$$D = \bigcup_{t=0}^{\infty} T^{-t}(\{x_2, \cdots, x_N\}) \tag{6}$$

and $h([0, 1]) = [0, 1]$. Moreover, the conclusions of Theorem 3.1 equally repeat in this case provided that $\frac{1}{2}$ is replaced by $\frac{1}{N}$ in the condition on the contraction parameter λ. Indeed, the only novelty sits in the number of t-preimages of discontinuity point $\{x_2, \cdots, x_N\}$ (where $h(x-0)-h(x) = \lambda^t$) which is now given by $(N - 1)N^t$.

If, in addition the map T is piecewise affine, then analogous to statements of Propositions 3.2 and 3.3 hold for arbitrary $N > 2$.

Next, we consider the case where the response g is also nonlinear, *i.e.* we assume

- relation (1) holds for some $0 < \lambda < 1$,
- the maps $g(\cdot, z)$ are all strictly increasing and the family is equicontinuous.

In such cases, the existence of a globally attracting synchronization function has previously been established [14, 17]. Here, we complete this result by specifying some properties of this function.

Proposition 4.1 *For any $f(x, y) = (\mathcal{T}(x, y), S(y))$ and $g(x, z)$ as above, there exists a function h whose graph $z = h(x)$ attracts all sequences $\{(x^t, z^t)\}$ of the skew-product (f, g). The function h has the following properties:*

- *it is right continuous,*
- *it is continuous in D^c and $h(x - 0) > h(x)$ if $x \in D$,*
- *it is nowhere monotonous,*
- *it has bounded variation if $\lambda < \frac{1}{N}$ and writes $h_c - h_d$ in this case, where h_c is strictly increasing and continuous and h_d is an increasing step function.*
- *it has infinite variation if $\lambda' > \frac{1}{N}$ where $\lambda' := \inf\limits_{x, z-z' \neq 0} \frac{|g(x,z)-g(x,z')|}{|z-z'|}$.*

Proof. Existence proofs in the nonlinear case have already been published in the literature [14, 17]. We provide another proof here that is more suitable to the present framework.

Let $g_x(y) := g(x,y)$ and $M = \sup_{x\in[0,1]} |g_x(0)| < \infty$. Relation (1) results in $|g_x(y)| \le |g_x(0)| + \lambda|y|$ which implies

$$g_x([-M_\lambda, M_\lambda]) \subset [-M_\lambda, M_\lambda], \quad \forall x \in [0,1]$$

where $M_\lambda = \frac{M}{1-\lambda}$.

Given an arbitrary pair (x,z) and $t \ge 0$, define the tth iterate $h_t(x,z)$ as follows

$$h_t(x,z) = g_{T(x)} \circ g_{T^2(x)} \circ \cdots \circ g_{T^t(x)}(z).$$

By refining the arguments above, one shows that the interval $[-M_\lambda, M_\lambda]$ is not only invariant for h_t but it is also absorbing. In particular every sequence $\{h_t(x,z)\}$ must be bounded.

Next choose two integers $t > s$. We have

$$|h_t(x,z) - h_s(x,z)| < \lambda^s |g_{T^{s+1}(x)} \circ \cdots \circ g_{T^t(x)}(z) - z|$$

which implies that $\{h_t(x,z)\}$ is a Cauchy sequence; hence the following limit exists for all $x \in [0,1]$

$$h(x) \equiv \lim_{t\to\infty} h_t(x,z)$$

and is independent of z. Moreover, the continuous dependence on z implies the following conjugacy equation, *i.e.*

$$\begin{aligned} g(x,h(x)) &= g_x(\lim_{t\to\infty} h_t(x,z)) \\ &= \lim_{t\to\infty} g_x(h_t(x,z)) = h \circ \mathcal{T}(x,y) \end{aligned}$$

for any $y \in [0,1]$. Global attraction easily follows by using contraction once again.

The arguments for properties of h are very similar to those in the linear case:

- Right continuity follows from both the fact that all $h_t(x,y)$ are right continuous and uniform convergence in the definition of h.

- The existence of left limit $h(x-0)$ and the continuity in D^c are obtained similarly by using also the monotonicity of the $g(\cdot,z)$. Now, if $T^{t_0}(x) \in D$, then by continuity outside D, we have

$$\begin{aligned} h(x-0) &= g_{T(x)} \circ g_{T^2(x)} \circ \cdots \circ g_{T^{t_0}(x)} \circ h(1) \\ h(x) &= g_{T(x)} \circ g_{T^2(x)} \circ \cdots \circ g_{T^{t_0}(x)} \circ h(0) \end{aligned}$$

  and thus $h(x-0) - h(x) > 0$ by strict monotonicity.

- The estimates on the bounded variation follow directly from the fact that if $T^{t_0}(x) \in D$, then

$$(\lambda')^{t_0}(h(1) - h(0)) \leq h(x - 0) - h(x) \leq \lambda^{t_0}(h(1) - h(0))$$

Finally the properties of the components h_c and h_d can be obtained in a similar way as in the proof of Theorem 3.1. □

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